

MAXIMAL REGULARITY FOR NON-AUTONOMOUS EVOLUTION EQUATIONS GOVERNED BY FORMS HAVING LESS REGULARITY

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ABSTRACT. We consider the maximal regularity problem for non-autonomous evolution equations

$$\begin{cases} u'(t) + A(t)u(t) &= f(t), \quad t \in (0, \tau] \\ u(0) &= u_0. \end{cases} \quad (0.1)$$

Each operator $A(t)$ is associated with a sesquilinear form $\mathfrak{a}(t)$ on a Hilbert space H . We assume that these forms all have the same domain V . It is proved in [11] that if the forms have some regularity with respect to t (e.g., piecewise α -Hölder continuous for some $\alpha > 1/2$) then the above problem has maximal L_p -regularity for all u_0 in the real-interpolation space $(H, \mathcal{D}(A(0)))_{1-1/p, p}$. In this paper we prove that the regularity required there can be improved for a class of sesquilinear forms. The forms considered here are such that the difference $\mathfrak{a}(t; \cdot, \cdot) - \mathfrak{a}(s; \cdot, \cdot)$ is continuous on a larger space than the common domain V . We give three examples which illustrate our results.

1. INTRODUCTION AND MAIN RESULTS

Let H and V be real or complex Hilbert spaces such that V is densely and continuously embedded in H . We denote by V' the (anti-)dual of V and by $[\cdot | \cdot]_H$ the scalar product of H and $\langle \cdot, \cdot \rangle$ the duality pairing $V' \times V$. The latter satisfies (as usual) $\langle v, h \rangle = [v | h]_H$ whenever $v \in H$ and $h \in V$. By the standard identification of H with H' we then obtain continuous and dense embeddings $V \hookrightarrow H \simeq H' \hookrightarrow V'$. We denote by $\|\cdot\|_V$ and $\|\cdot\|_H$ the norms of V and H , respectively.

We consider the non-autonomous evolution equation

$$\begin{cases} u'(t) + A(t)u(t) &= f(t), \quad t \in (0, \tau] \\ u(0) &= u_0, \end{cases} \quad (\text{P})$$

where each operator $A(t)$, $t \in [0, \tau]$, is associated with a sesquilinear form $\mathfrak{a}(t)$. We assume that $t \mapsto \mathfrak{a}(t; u, v)$ is measurable for all $u, v \in V$ and

[H1] (constant form domain) $\mathcal{D}(\mathfrak{a}(t)) = V$.

[H2] (uniform boundedness) there exists $M > 0$ such that for all $t \in [0, \tau]$ and $u, v \in V$, we have $|\mathfrak{a}(t; u, v)| \leq M\|u\|_V\|v\|_V$.

[H3] (uniform quasi-coercivity) there exist $\alpha_1 > 0$, $\delta \in \mathbb{R}$ such that for all $t \in [0, \tau]$ and all $u, v \in V$ we have $\alpha_1\|u\|_V^2 \leq \operatorname{Re}\mathfrak{a}(t; u, u) + \delta\|u\|_H^2$.

For each t , we can associate with the form $\mathfrak{a}(t; \cdot, \cdot)$ an operator $A(t)$ defined as follows

$$\begin{aligned} \mathcal{D}(A(t)) &= \{u \in V, \exists v \in H : \mathfrak{a}(t, u, \varphi) = [v | \varphi]_H \quad \forall \varphi \in V\} \\ A(t)u &:= v. \end{aligned}$$

On the other hand, there exists a linear operator $\mathcal{A}(t) : V \rightarrow V'$ such that $\mathfrak{a}(t; u, v) = \langle \mathcal{A}(t)u, v \rangle$ for all $u, v \in V$. The operator $\mathcal{A}(t)$ can be seen as an unbounded operator on V' with domain V and $A(t)$ is the part of $\mathcal{A}(t)$ on H , that is,

$$\mathcal{D}(A(t)) = \{u \in V, \mathcal{A}(t)u \in H\}, \quad A(t)u = \mathcal{A}(t)u.$$

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It is a known fact that $-A(t)$ and $-\mathcal{A}(t)$ both generate holomorphic semigroups $(e^{-sA(t)})_{s \geq 0}$ and $(e^{-s\mathcal{A}(t)})_{s \geq 0}$ on H and V' , respectively. For each $s \geq 0$, $e^{-sA(t)}$ is the restriction of $e^{-s\mathcal{A}(t)}$ to H . For all this, we refer to Ouhabaz [15, Chapter 1].

The notion of maximal L_p -regularity for the above Cauchy problem is defined as follows.

Definition 1.1. Fix $u_0 \in H$. We say that (P) has maximal L_p -regularity (in H) if for each $f \in L_p(0, \tau; H)$ there exists a unique $u \in W_p^1(0, \tau; H)$, such that $u(t) \in \mathcal{D}(A(t))$ for almost all t , which satisfies (P) in the L_p -sense.

Recall that under the assumptions [H1]–[H3], J.L. Lions proved maximal L_2 -regularity in V' for all initial data $u_0 \in H$, see e.g. [12], [17, page 112]. This means that for every $u_0 \in H$ and $f \in L_2(0, \tau; V')$, the equation

$$\begin{cases} u'(t) + \mathcal{A}(t)u(t) &= f(t) \\ u(0) &= u_0 \end{cases} \quad (\text{P}')$$

has a unique solution $u \in W_2^1(0, \tau; V') \cap L_2(0, \tau; V)$. It is a remarkable fact that Lions's theorem does not require any regularity assumption (with respect to t) on the sesquilinear forms apart from measurability. Note however that maximal regularity in H differs considerably from maximal regularity in V' . The fact that the forms have the same domain means that the operators $\mathcal{A}(t)$ have constant domains in V' and this fact plays an important role in proving maximal regularity. The operators $A(t)$ may have different domains as operators on H . The problem of maximal regularity in H for (P) was stated explicitly by Lions and it is still open in general. Some progress has been made in recent years.

First, recall that Bardos [6] proved maximal L_2 -regularity in H with initial data $u_0 \in V$ provided $\mathcal{D}(A(t)^{1/2}) = V$ as space and topologically and assuming that $t \mapsto \mathfrak{a}(t; u, v)$ is C^1 on $[0, \tau]$. His result was extended in Arendt et al. [4] for Lipschitz forms (with respect to t) and allowing a multiplicative perturbation by bounded operators $B(t)$ which are measurable in t . The maximal L_2 -regularity is then proved for the evolution problem associated with $B(t)A(t)$. Ouhabaz and Spina [16] proved maximal L_p -regularity on H for all $p \in (1, \infty)$ under the assumption that $t \mapsto \mathfrak{a}(t; u, v)$ is α -Hölder continuous for some $\alpha > 1/2$. The result in [16] concerns the problem (P) with initial data $u(0) = 0$. A simple example was given recently by Dier [9] which shows that in general the answer to Lions' problem is negative. The following positive result was proved by Haak and Ouhabaz [11].

Theorem 1.2. Suppose that the forms $(\mathfrak{a}(t))_{0 \leq t \leq \tau}$ satisfy the hypotheses [H1]–[H3] and the regularity condition

$$|\mathfrak{a}(t; u, v) - \mathfrak{a}(s; u, v)| \leq \omega(|t-s|) \|u\|_V \|v\|_V \quad (1.1)$$

where $\omega : [0, \tau] \rightarrow [0, \infty)$ is a non-decreasing function such that

$$\int_0^\tau \frac{\omega(t)}{t^{3/2}} dt < \infty. \quad (1.2)$$

Then the Cauchy problem (P) with $u_0 = 0$ has maximal L_p -regularity in H for all $p \in (1, \infty)$. If in addition ω satisfies the p -Dini condition

$$\int_0^\tau \left(\frac{\omega(t)}{t} \right)^p dt < \infty, \quad (1.3)$$

then (P) has maximal L_p -regularity for all $u_0 \in (H, \mathcal{D}(A(0)))_{1-1/p, p}$. Moreover there exists a positive constant C such that

$$\begin{aligned} & \|u\|_{L_p(0, \tau; H)} + \|u'\|_{L_p(0, \tau; H)} + \|A(\cdot)u(\cdot)\|_{L_p(0, \tau; H)} \\ & \leq C \left[\|f\|_{L_p(0, \tau; H)} + \|u_0\|_{(H, \mathcal{D}(A(0)))_{1-1/p, p}} \right]. \end{aligned}$$

In this theorem, $(H, \mathcal{D}(A(0)))_{1-1/p, p}$ denotes the classical real-interpolation space, see [18, Chapter 1.13] or [13, Proposition 6.2].

In the case where $p = 2$, we obtain maximal L_2 -regularity for $u(0) \in \mathcal{D}((\delta + A(0))^{1/2})$. The theorem can be used in the case where $t \mapsto \mathbf{a}(t; u, v)$ is α -Hölder continuous for some $\alpha > \frac{1}{2}$. The case of piecewise α -Hölder continuous is also covered. See [11] for the details.

The aim of the present paper is to weaken the regularity assumption measured by (1.2) and (1.3) in some situations. More precisely, we assume in addition to [H1]–[H3] that there exist $\beta, \gamma \in [0, 1]$ such that for all $u, v \in V$

$$|\mathbf{a}(t; u, v) - \mathbf{a}(s; u, v)| \leq \omega(|t-s|) \|u\|_{V_\beta} \|v\|_{V_\gamma}, \quad (1.4)$$

where $V_\beta := [H, V]_\beta$ is the classical complex interpolation space for $\beta \in [0, 1]$ with $V_0 = H$ and $V_1 = V$. If $\beta, \gamma \in (0, 1)$, the assumption (1.4) means that the difference of the forms is defined on a larger space than the common form domain V .

Our main result is the following.

Theorem 1.3. *Suppose that the forms $(\mathbf{a}(t))_{0 \leq t \leq \tau}$ satisfy the hypotheses [H1]–[H3] and (1.4) where $\omega : [0, \tau] \rightarrow [0, \infty)$ is a non-decreasing function such that*

$$\int_0^\tau \frac{\omega(t)}{t^{1+\frac{\gamma}{2}}} dt < \infty. \quad (1.5)$$

Then the Cauchy problem (P) with $u_0 = 0$ has maximal L_p -regularity in H for all $p \in (1, \infty)$. If in addition ω satisfies the p -Dini condition

$$\int_0^\tau \left(\frac{\omega(t)}{t^{\frac{\beta+\gamma}{2}}} \right)^p dt < \infty, \quad (1.6)$$

then (P) has maximal L_p -regularity for all $u_0 \in (H, \mathcal{D}(A(0)))_{1-1/p, p}$. Moreover there exists a positive constant C such that

$$\begin{aligned} & \|u\|_{L_p(0, \tau; H)} + \|u'\|_{L_p(0, \tau; H)} + \|A(\cdot)u(\cdot)\|_{L_p(0, \tau; H)} \\ & \leq C \left[\|f\|_{L_p(0, \tau; H)} + \|u_0\|_{(H, \mathcal{D}(A(0)))_{1-1/p, p}} \right]. \end{aligned}$$

A related result was proved recently by Arendt and Monniaux [5] who prove maximal L_2 -regularity under the additional assumptions that the Kato square root property $V = \mathcal{D}(A(0)^{1/2})$ holds, $\beta = \gamma$ in (1.4) and an additional growth condition $\omega(t) \leq Ct^{\frac{\gamma}{2}}$. We observe that in our result β does not come into play if $u_0 = 0$. We expect the theorem to be true with $\min(\beta, \gamma)$ in place of γ in (1.5). The following two corollaries follow immediately from the theorem.

Corollary 1.4. *Suppose that the forms $(\mathbf{a}(t))_{0 \leq t \leq \tau}$ satisfy the hypotheses [H1]–[H3] and α -Hölder continuous in the sense that*

$$|\mathbf{a}(t, u, v) - \mathbf{a}(s, u, v)| \leq C|t-s|^\alpha \|u\|_{V_\beta} \|v\|_{V_\gamma}. \quad (1.7)$$

Then the Cauchy problem (P) with $u_0 = 0$ has maximal L_p -regularity in H for all $p \in (1, \infty)$ provided $\alpha > \frac{\gamma}{2}$. If in addition $\alpha > \frac{\beta+\gamma}{2} - \frac{1}{p}$, then (P) has maximal L_p -regularity for all $u_0 \in (H, \mathcal{D}(A(0)))_{1-1/p, p}$. Moreover there exists a positive constant C such that

$$\begin{aligned} & \|u\|_{L_p(0, \tau; H)} + \|u'\|_{L_p(0, \tau; H)} + \|A(\cdot)u(\cdot)\|_{L_p(0, \tau; H)} \\ & \leq C \left[\|f\|_{L_p(0, \tau; H)} + \|u_0\|_{(H, \mathcal{D}(A(0)))_{1-1/p, p}} \right]. \end{aligned}$$

Corollary 1.5. *Suppose that the forms $(\mathbf{a}(t))_{0 \leq t \leq \tau}$ satisfy the hypotheses [H1]–[H3] and α -Hölder continuous in the sense that*

$$|\mathbf{a}(t, u, v) - \mathbf{a}(s, u, v)| \leq C|t-s|^\alpha \|u\|_{V_\beta} \|v\|_{V_\gamma}, \quad (1.8)$$

for some $\alpha > \frac{\gamma}{2}$. Then the Cauchy problem (P) has maximal L_2 -regularity in H for all $u_0 \in \mathcal{D}((\delta + A(0))^{1/2})$. Moreover there exists a positive constant C such that

$$\begin{aligned} & \|u\|_{L_p(0, \tau; H)} + \|u'\|_{L_p(0, \tau; H)} + \|A(\cdot)u(\cdot)\|_{L_p(0, \tau; H)} \\ & \leq C \left[\|f\|_{L_p(0, \tau; H)} + \|(\delta + A(0))^{1/2} u_0\|_H \right]. \end{aligned}$$

Notation: We shall often use C or C' to denote all inessential constants. We use $W_p^1(0, \tau; E)$ as well as $H^s(\Omega) := W_2^s(\Omega)$ for the classical Sobolev spaces. The first one is the Sobolev space of order one of L_p -functions on $(0, \tau)$ with values in a Banach space E and the second one is the Sobolev space of order s of L_2 scalar-valued functions acting on a domain Ω .

2. PROOF OF THE MAIN RESULT

Throughout this section we adopt the notation of the introduction. We shall use the strategy and ideas of proof of Theorem 1.2 in [11] with some modifications in order to incorporate the additional assumption (1.4).

Recall that the solution u to (P) exists in V' by Lions' theorem mentioned in the introduction. The aim is to prove that $u(t) \in \mathcal{D}(A(t))$ for almost all $t \in [0, \tau]$ and $A(\cdot)u(\cdot) \in L_p(0, \tau; H)$. From this and the Cauchy problem (P) it follows that $u \in W_p^1(0, \tau; H)$.

From now on we assume without loss of generality that the forms are coercive, that is [H3] holds with $\delta = 0$. The reason is that by replacing $A(t)$ by $A(t) + \delta$, the solution v of (P) is $v(t) = u(t)e^{-\delta t}$ and it is clear that $u \in W_p^1(0, \tau; H)$ if and only if $v \in W_p^1(0, \tau; H)$.

First we have the representation formula (see [11] for all what follows)

$$\begin{aligned} u(t) &= \int_0^t e^{-(t-s)A(t)} (\mathcal{A}(t) - \mathcal{A}(s)) u(s) ds \\ &\quad + \int_0^t e^{-(t-s)A(t)} f(s) ds + e^{-tA(t)} u_0. \end{aligned} \quad (2.1)$$

In addition,

$$\mathcal{A}(t)u(t) = (Q\mathcal{A}(\cdot)u(\cdot))(t) + (Lf)(t) + (Ru_0)(t), \quad (2.2)$$

where

$$\begin{aligned} (Qg)(t) &:= \int_0^t \mathcal{A}(t) e^{-(t-s)A(t)} (\mathcal{A}(t) - \mathcal{A}(s)) \mathcal{A}(s)^{-1} g(s) ds \\ (Lg)(t) &:= \mathcal{A}(t) \int_0^t e^{-(t-s)A(t)} g(s) ds \quad \text{and} \quad (Ru_0)(t) := \mathcal{A}(t) e^{-tA(t)} u_0. \end{aligned}$$

The aim is to prove boundedness on $L_p(0, \tau; H)$ of the operators L , R and Q and then by a simple scaling argument the norm of Q is less than 1. This allows us to invert $(I - Q)$ on $L_p(0, \tau; H)$ and conclude from (2.2) that $A(\cdot)u(\cdot) \in L_p(0, \tau; H)$.

We start with the operator L . The following result is Lemma 2.6 in [11].

Lemma 2.1. *Suppose that in addition to the assumptions [H1]- [H3] that (1.4) holds for some $\beta, \gamma \in [0, 1]$ and $\omega : [0, \tau] \rightarrow [0, \infty)$ a non-decreasing function such that*

$$\int_0^\tau \frac{\omega(t)^2}{t} dt < \infty. \quad (2.3)$$

Then L is a bounded operator on $L_p(0, \tau; H)$ for all $p \in (1, \infty)$.

Now we deal with the operator R .

Recall first that $-A(t)$ is the generator of a bounded holomorphic semigroup of angle $\frac{\pi}{2} - \arctan(\frac{M}{\alpha_0})$ where α_0 and M are as in the assumptions [H2] and [H3]. See [15, Chapter 1] or [11]. In addition we have

Lemma 2.2. *Let $\omega : \mathbb{R} \rightarrow \mathbb{R}_+$ be some function and assume that*

$$|\mathbf{a}(t; u, v) - \mathbf{a}(s; u, v)| \leq \omega(|t-s|) \|u\|_{V_\beta} \|v\|_{V_\gamma}$$

for all $u, v \in V$. Then

$$\|R(z, A(t)) - R(z, A(s))\|_{\mathcal{B}(H)} \leq \frac{c_\theta}{|z|^{1-\frac{\beta+\gamma}{2}}} \omega(|t-s|)$$

for all $z \notin S_\theta$ with any fixed $\theta > \arctan(M/\alpha)$. The constant c_θ is independent of z, t and s .

Proof. Fix $\theta > \arctan(M/\alpha)$. Note that (see [11], Proposition 2.1 d))

$$\|(z - A(t))^{-1}x\|_V \leq \frac{C_\theta}{\sqrt{|z|}} \|x\|_H \text{ for all } z \notin S_\theta. \quad (2.4)$$

Observe that for $u, v \in V$,

$$\begin{aligned} & |[R(z, A(t))u - R(z, A(s))u]v|_H| \\ &= |[R(z, A(t))(A(s) - A(t))R(z, A(s))u]v|_H| \\ &= |[A(s)R(z, A(s))u]R(z, A(t))^*v|_H - [A(t)R(z, A(s))u]R(z, A(t))^*v|_H| \\ &= |\mathfrak{a}(s; R(z, A(s))u, R(z, A(t))^*v) - \mathfrak{a}(t; R(z, A(s))u, R(z, A(t))^*v)| \\ &\leq \omega(|t-s|) \|R(z, A(s))u\|_{V_\beta} \|R(z, A(t))^*v\|_{V_\gamma} \\ &\leq \frac{C_\theta}{|z|^{2-\frac{\beta+\gamma}{2}}} \omega(|t-s|) \|u\|_H \|v\|_H, \end{aligned}$$

where we used (2.4) and a standard interpolation argument. \square

Lemma 2.3. *Assume (1.6). Then there exists $C > 0$ such that*

$$\|Ru_0\|_{L_p(0, \tau; H)} \leq C \|u_0\|_{(H, \mathcal{D}(A(0)))_{1-1/p, p}},$$

for all $u_0 \in (H, \mathcal{D}(A(0)))_{1-1/p, p}$.

Proof. Recall that the operator R is given by $(Rg)(t) = A(t)e^{-tA(t)}g$ for $g \in H$. Let

$$(R_0g)(t) := A(0)e^{-tA(0)}g.$$

We estimate the difference $(R - R_0)g$. Let $v \in H$ and $\Gamma = \partial S_\theta$ with $\theta < \pi/2$ as in (2.4). Then the functional calculus for the sectorial operators $A(t)$ and $A(0)$ gives

$$\begin{aligned} & \left[A(t)e^{-tA(t)}g - A(0)e^{-tA(0)}g \right] v \Big|_H \\ &= \frac{1}{2\pi i} \int_\Gamma [ze^{-tz} [R(z, A(t)) - R(z, A(0))]g] v \Big|_H dz \\ &= \frac{1}{2\pi i} \int_\Gamma [ze^{-tz} R(z, A(t)) [\mathcal{A}(0) - \mathcal{A}(t)] R(z, A(0))g] v \Big|_H dz \\ &= \frac{1}{2\pi i} \int_\Gamma [ze^{-tz} [\mathcal{A}(0) - \mathcal{A}(t)] R(z, A(0))g] R(z, A(t))^*v \Big|_H dz \\ &= \frac{1}{2\pi i} \int_\Gamma ze^{-tz} [\mathfrak{a}(0; R(z, A(0))g, R(z, A(t))^*v) - \\ &\quad \mathfrak{a}(t; R(z, A(0))g, R(z, A(t))^*v)] dz. \end{aligned}$$

It follows from (1.4) and Lemma 2.2 that

$$\begin{aligned} & |[(Rg - R_0g)(t)] v|_H| \\ &\leq \frac{1}{2\pi} \int_\Gamma \omega(t) |z| e^{-t \operatorname{Re}(z)} \|R(z, A(0))g\|_{V_\beta} \|R(z, A(t))^*v\|_{V_\gamma} |dz| \\ &\leq C \omega(t) \|g\|_H \|v\|_H \int_\Gamma |z|^{\frac{\beta+\gamma}{2}-1} e^{-t \operatorname{Re} z} |dz| \\ &\leq C' \frac{\omega(t)}{t^{\frac{\beta+\gamma}{2}}} \|g\|_H \|v\|_H. \end{aligned}$$

Since this true for all $v \in H$ we conclude that

$$\|(Ru_0)(t) - (R_0u_0)(t)\|_H \leq C' \frac{\omega(t)}{t^{\frac{\beta+\gamma}{2}}} \|u_0\|_H. \quad (2.5)$$

From the hypothesis (1.6) it follows that $Ru_0 - R_0u_0 \in L_p(0, \tau; H)$. On the other hand, since $A(0)$ is invertible, it is well-known that $A(0)e^{-tA(0)}u_0 \in L_p(0, \tau; H)$ if and only if $u_0 \in (H, \mathcal{D}(A(0)))_{1-1/p, p}$ (see Triebel [18, Theorem 1.14]). Therefore, $Ru_0 \in L_p(0, \tau; H)$ and the lemma is proved. \square

Proof of Theorem 1.3. As we already mentioned before, the arguments are essentially the same as in [11] in which we use the additional assumption (1.4) to weaken the required regularity on the forms. We start with the case $u_0 = 0$ and let $f \in C_c^\infty(0, \tau; H)$. From (2.2) we have

$$(I - Q)A(\cdot)u(\cdot) = Lf(\cdot). \quad (2.6)$$

Recall that L is bounded on $L_p(0, \tau; H)$ by Lemma 2.1. We shall now prove that Q is bounded on $L_p(0, \tau; H)$. Let $g \in L_2(0, \tau; H)$ and $v \in H$. We have

$$\begin{aligned} & |[Qg(t) | v]_H| \\ &= \int_0^t [\mathfrak{a}(t; \mathcal{A}(s)^{-1}g(s), \mathcal{A}(t)^* e^{-(t-s)\mathcal{A}(t)^*}v) - \\ & \quad \mathfrak{a}(s; \mathcal{A}(s)^{-1}g(s), \mathcal{A}(t)^* e^{-(t-s)\mathcal{A}(t)^*}v)] \, ds \end{aligned} \quad (2.7)$$

$$\leq \int_0^t \omega(|t-s|) \|\mathcal{A}(s)^{-1}g(s)\|_{V_\beta} \|\mathcal{A}(t)^* e^{-(t-s)\mathcal{A}(t)^*}v\|_{V_\gamma} \, ds. \quad (2.8)$$

By coercivity assumption one has easily

$$\|\mathcal{A}(t)e^{-s\mathcal{A}(t)}v\|_V \leq \frac{C}{s^{\frac{\gamma}{2}}} \|v\|_H$$

(see Proposition 2.1 c) in [11]). Hence by interpolation

$$\|\mathcal{A}(t)e^{-s\mathcal{A}(t)}v\|_{V_\gamma} \leq \frac{C}{s^{1+\frac{\gamma}{2}}} \|v\|_H. \quad (2.9)$$

The constant C is independent of t, s and v . The adjoint operators $\mathcal{A}(t)^*$ satisfy the same estimates. Now we estimate $\|\mathcal{A}(s)^{-1}g(s)\|_{V_\beta}$. By coercivity

$$\begin{aligned} \|\mathcal{A}(s)^{-1}g(s)\|_{V_\beta}^2 &\leq C \|\mathcal{A}(s)^{-1}g(s)\|_V^2 \\ &\leq \frac{C}{\alpha_0} \operatorname{Re} \langle \mathcal{A}(s)^{-1}g(s), \mathcal{A}(s)^{-1}g(s) \rangle \\ &= \frac{C}{\alpha_0} \operatorname{Re} \langle \mathcal{A}(s) \mathcal{A}(s)^{-1}g(s), \mathcal{A}(s)^{-1}g(s) \rangle \\ &= \frac{C}{\alpha_0} \operatorname{Re} [g(s) | \mathcal{A}(s)^{-1}g(s)]_H \\ &\leq \frac{C}{\alpha_0} \|g(s)\|_H^2 \|\mathcal{A}(s)^{-1}\|_{\mathcal{B}(H)}. \end{aligned}$$

Inserting this and (2.9) (for the adjoint operators) in (2.8) we obtain

$$\|(Qg)(t)\|_H \leq \int_0^t \frac{C'}{(t-s)^{1+\gamma/2}} \omega(t-s) \|\mathcal{A}(s)^{-1}\|_{\mathcal{B}(H)}^{1/2} \|g(s)\|_H \, ds. \quad (2.10)$$

Now, once we replace $A(s)$ by $A(s) + \mu$, (2.9) is valid with a constant independent of $\mu \geq 0$ and using the estimate

$$\|(\mathcal{A}(s) + \mu)^{-1}\|_{\mathcal{B}(H)} \leq \frac{1}{\mu},$$

in (2.10) for $A(s) + \mu$ we see that

$$\|(Qg)(t)\|_H \leq \frac{C'}{\sqrt{\mu}} \int_0^t \frac{\omega(t-s)}{(t-s)^{1+\gamma/2}} \|g(s)\|_H \, ds.$$

The operator S defined by

$$Sh(t) := \int_0^t \frac{\omega(t-s)}{(t-s)^{1+\gamma/2}} h(s) \, ds$$

is bounded on $L_p(0, \tau; \mathbb{R})$ since it has a kernel $\omega(t-s)(t-s)^{1+\gamma/2}$ which is integrable with respect to each variable uniformly with respect to the other variable by (1.5). It follows that Q is bounded on $L_p(0, \tau; H)$ with norm of at most $\frac{C''}{\sqrt{\mu}}$ for some constant C'' . Taking then μ large enough makes Q strictly contractive such that $(I - Q)^{-1}$ is bounded on $L_p(0, \tau; H)$. Then, for $f \in C_c^\infty(0, \tau; H)$, (2.6) can be rewritten as

$$A(\cdot)u(\cdot) = (I - Q)^{-1}Lf(\cdot).$$

This shows that $u(t) \in \mathcal{D}(A(t))$ for almost t and $A(\cdot)u(\cdot) \in L_p(0, \tau; H)$.

For general $u_0 \in (H, \mathcal{D}(A(0)))_{1-1/p, p}$ we suppose in addition to (1.5) that (1.6) holds. Lemma 2.3 shows that $Ru_0 \in L_p(0, \tau; H)$. As previously we conclude that

$$A(\cdot)u(\cdot) = (I - Q)^{-1}(Lf + Ru_0),$$

whenever $f \in C_c^\infty(0, \tau; H)$. Thus taking the L_p norm yields

$$\|A(\cdot)u(\cdot)\|_{L_p(0, \tau; H)} \leq C\|(Lf + Ru_0)\|_{L_p(0, \tau; H)}.$$

We use again the previous estimates on L and R to obtain

$$\|A(\cdot)u(\cdot)\|_{L_p(0, \tau; H)} \leq C' \left[\|f\|_{L_p(0, \tau; H)} + \|u_0\|_{(H, \mathcal{D}(A(0)))_{1-1/p, p}} \right].$$

Using the equation (P) we obtain a similar estimate for u' and so

$$\begin{aligned} & \|u'(\cdot)\|_{L_p(0, \tau; H)} + \|A(\cdot)u(\cdot)\|_{L_p(0, \tau; H)} \\ & \leq C'' \left[\|f\|_{L_p(0, \tau; H)} + \|u_0\|_{(H, \mathcal{D}(A(0)))_{1-1/p, p}} \right]. \end{aligned}$$

We write $u(t) = A(t)^{-1}A(t)u(t)$ and use one again the fact that the norms of $A(t)^{-1}$ on H are uniformly bounded we obtain

$$\begin{aligned} & \|u(t)\|_{L_p(0, \tau; H)} \leq C_1 \|A(\cdot)u(\cdot)\|_{L_p(0, \tau; H)} \\ & \leq C_2 \left[\|f\|_{L_p(0, \tau; H)} + \|u_0\|_{(H, \mathcal{D}(A(0)))_{1-1/p, p}} \right]. \end{aligned}$$

We conclude therefore that the following a priori estimate holds

$$\begin{aligned} & \|u\|_{L_p(0, \tau; H)} + \|u'\|_{L_p(0, \tau; H)} + \|A(\cdot)u(\cdot)\|_{L_p(0, \tau; H)} \\ & \leq C \left[\|f\|_{L_p(0, \tau; H)} + \|u_0\|_{(H, \mathcal{D}(A(0)))_{1-1/p, p}} \right], \end{aligned} \quad (2.11)$$

where the constant C does not depend on $f \in C_c^\infty(0, \tau; H)$.

The latter estimate extends by density to all $f \in L_p(0, \tau; H)$ (see [11]). This proves the desired maximal L_p -regularity property. \square

3. EXAMPLES

Schrödinger operators with time dependent potentials.

We consider on $H = L^2(\mathbb{R}^d)$ Schrödinger operators $A(t) = -\Delta + m(t, \cdot)$ with time dependent potentials $m(t, x)$. We make the following assumptions:

- There exists a non-negative function $m_0 \in L_{1, loc}$ and two positive constants c_1, c_2 such that

$$c_1 m_0(x) \leq m(t, x) \leq c_2 m_0(x), \quad x \in \mathbb{R}^d, \quad t \in [0, \tau]. \quad (3.1)$$

- There exists a function $p_0 \in L_{1, loc}$ such that

$$|m(t, x) - m(s, x)| \leq |t - s|^\alpha p_0(x), \quad x \in \mathbb{R}^d, \quad t, s \in [0, \tau]. \quad (3.2)$$

- There exists $C > 0$ and $s \in [0, 1]$ such that

$$\int_{\mathbb{R}^d} p_0(x) |u(x)|^2 dx \leq C \|u\|_{H^s(\mathbb{R}^d)}, \quad u \in C_c^\infty. \quad (3.3)$$

Note that assumption (3.3) is satisfied for several weights p_0 . For example, this is the case for $p_0 = \frac{1}{|x|^2}$ and $s = 1$ by Hardy's inequality. On the other hand, by Hölder's inequality and classical Sobolev embeddings for H^s one finds r_s such that (3.3) holds for $p_0 \in L_{r_s}$. Obviously, (3.3) holds with $s = 0$ if $p_0 \in L_\infty$.

The operator $A(t) = -\Delta + m(t, x)$ is defined as the operator associated with the form

$$\mathfrak{a}(t; u, v) = \int_{\mathbb{R}^d} \nabla u \cdot \nabla v dx + \int_{\mathbb{R}^d} m(t, \cdot) uv dx$$

defined on

$$V = \{u \in H^1(\mathbb{R}^d), \int_{\mathbb{R}^d} m_0 |u|^2 dx < \infty\}.$$

The forms $\mathbf{a}(t; \cdot, \cdot)$ satisfy the standard assumptions [H1]–[H3]. Using the additional assumption (3.3) we can estimate the difference $\mathbf{a}(t; u, v) - \mathbf{a}(s; u, v)$ as follows

$$\begin{aligned} |\mathbf{a}(t; u, v) - \mathbf{a}(s; u, v)| &= \left| \int_{\mathbb{R}^d} [m(t, \cdot) - m(s, \cdot)] uv \, dx \right| \\ &\leq |t - s|^\alpha \int_{\mathbb{R}^d} p_0(x) |uv| \, dx \\ &\leq |t - s|^\alpha \left(\int_{\mathbb{R}^d} p_0(x) |u|^2 \, dx \right)^{1/2} \left(\int_{\mathbb{R}^d} p_0(x) |v|^2 \, dx \right)^{1/2} \\ &\leq C |t - s|^\alpha \|u\|_{H^s(\mathbb{R}^d)} \|v\|_{H^s(\mathbb{R}^d)}. \end{aligned}$$

Therefore, we can apply Theorem 1.3 to obtain maximal L_p -regularity for the evolution equation associated with $A(t) = -\Delta + m(t, \cdot)$ under the condition $\alpha > s/2$ where α and s are as in (3.2) and (3.3). For $p = 2$, the initial data u_0 can be taken in $V = \mathcal{D}(A(0)^{1/2})$. For $p \neq 2$ we assume $u_0 \in (H, \mathcal{D}(A(0)))_{1-1/p, p}$ and $\alpha > \max(s/2, s - 1/p)$ by condition (1.6).

Elliptic operators with Robin boundary conditions.

Let Ω be a bounded domain of \mathbb{R}^d with Lipschitz boundary $\partial\Omega$. We denote by Tr the classical trace operator. Let $\beta : [0, \tau] \times \partial\Omega \rightarrow [0, \infty)$ and $a_k : [0, \tau] \times \Omega \rightarrow \mathbb{R}$ be bounded measurable functions for $k = 1, \dots, d$ such that

$$|\beta(t, x) - \beta(s, x)| \leq C |t - s|^\alpha, \quad t, s \in [0, \tau], x \in \partial\Omega$$

and

$$|a_k(t, x) - a_k(s, x)| \leq C |t - s|^\alpha, \quad t, s \in [0, \tau], x \in \Omega.$$

We define the form

$$\mathbf{a}(t; u, v) := \int_{\Omega} \nabla u \cdot \nabla v \, dx + \sum_{k=1}^d \int_{\Omega} a_k(t, x) \partial_k u \cdot v \, dx + \int_{\partial\Omega} \beta(t, \cdot) \text{Tr}(u) \text{Tr}(v) \, d\sigma,$$

for all $u, v \in H^1(\Omega)$. The associated operator $A(t)$ is formally given by

$$A(t) = -\Delta + \sum_{k=1}^d a_k(t, x) \partial_k u$$

and subject to the time dependent Robin boundary condition:

$$\frac{\partial u}{\partial n} + \beta(t, \cdot) u = 0 \text{ on } \partial\Omega.$$

Here $\frac{\partial u}{\partial n}$ denotes the normal derivative.

Now we check (1.4). We have for $u, v \in H^1(\Omega)$,

$$\begin{aligned} |\mathbf{a}(t; u, v) - \mathbf{a}(s; u, v)| &= \left| \sum_{k=1}^d \int_{\Omega} [a_k(t, \cdot) - a_k(s, \cdot)] \partial_k u \cdot v \, dx + \int_{\partial\Omega} [\beta(t, \cdot) - \beta(s, \cdot)] \text{Tr}(u) \text{Tr}(v) \, d\sigma \right| \\ &\leq C |t - s|^\alpha \left(\|u\|_{H^1(\Omega)} + \|u\|_{H^{1/2}(\Omega)} \|v\|_{H^{1/2}(\Omega)} \right), \end{aligned}$$

where we used the fact that the trace operator is bounded from $H^{1/2}(\Omega)$ into $L_2(\partial\Omega)$. Hence

$$|\mathbf{a}(t; u, v) - \mathbf{a}(s; u, v)| \leq C |t - s|^\alpha \|u\|_{H^1(\Omega)} \|v\|_{H^{1/2}(\Omega)}.$$

We apply Theorem 1.3 or the subsequent corollaries to obtain maximal L_2 -regularity for the corresponding evolution equation under the condition $\alpha > 1/4$ for initial data $u(0) \in H^1(\Omega) = \mathcal{D}(A(0)^{1/2})$. We also have maximal L_p -regularity for $1 < p < \infty$ if $\alpha > \max(\frac{1}{4}, \frac{3}{4} - \frac{1}{p})$ and $u(0) \in (H, \mathcal{D}(A(0)))_{1-1/p, p}$. In the case $p = 2$ and $a_k = 0$, this result was proved in [5].

Elliptic operators with Wentzell boundary conditions.

We wish to consider the heat equation with time dependent Wentzell boundary conditions:

$$\beta(t, \cdot)u + \frac{\partial u}{\partial n} + \Delta u = 0 \text{ on } \partial\Omega. \quad (3.4)$$

As in the previous example, we assume that Ω is a bounded Lipschitz domain and $\beta : [0, \tau] \times \partial\Omega \rightarrow [0, \infty)$ is a bounded measurable function such that

$$|\beta(t, x) - \beta(s, x)| \leq C|t - s|^\alpha, \quad t, s \in [0, \tau], x \in \partial\Omega.$$

In order to consider the Laplacian with Wentzell boundary conditions it is convenient to work on $H := L_2(\Omega) \oplus L_2(\partial\Omega)$ (see [3] or [10]). Set

$$V = \{(u, \text{Tr}(u)), u \in H^1(\Omega)\}$$

and define the form

$$\mathbf{a}(t; (u, \text{Tr}(u)), (v, \text{Tr}(v))) = \int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} \beta(t, \cdot) \text{Tr}(u) \text{Tr}(v) \, d\sigma,$$

for $u, v \in H^1(\Omega)$. The forms $\mathbf{a}(t)$ are well defined on V and satisfy the assumptions [H1]–[H3]. In addition,

$$\begin{aligned} & |\mathbf{a}(t; (u, \text{Tr}(u)), (v, \text{Tr}(v))) - \mathbf{a}(s; (u, \text{Tr}(u)), (v, \text{Tr}(v)))| \\ & \leq \int_{\partial\Omega} |\beta(t, \cdot) - \beta(s, \cdot)| \text{Tr}(u) \text{Tr}(v) \, d\sigma \\ & \leq C|t - s|^\alpha \|\text{Tr}(u)\|_{L_2(\partial\Omega)} \|\text{Tr}(v)\|_{L_2(\partial\Omega)} \\ & \leq C|t - s|^\alpha \|(u, \text{Tr}(u))\|_H \|(v, \text{Tr}(v))\|_H. \end{aligned}$$

We apply again Theorem 1.3 and obtain maximal L_p -regularity on $L_2(\Omega) \oplus L_2(\partial\Omega)$ for all $p \in (1, \infty)$ and $u(0) \in H^1(\Omega)$ under the sole condition that $\alpha > 0$.

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